



D1.2. Formulation of multiple scattering

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D.1.2: Formulation of multiple scattering

WP1. T1.2. Multiple scattering formulation for the previous objects.

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4. Introduction

4.1. Executive Summary

The following document presents a comprehensive exploration of multiple scattering theory and its applications in the analysis of complex structural arrangements. Multiple scattering theory stands as a versatile analytical framework, offering insights into the behaviour of clusters of scatterers subjected to external fields. This document delves into its theoretical underpinnings, practical methodologies, and real-world applications, aiming to elucidate its significance in various scientific and engineering disciplines.

Beginning with an overview of the fundamental principles of multiple scattering theory, we delve into its mathematical formulations and underlying assumptions. We explore how this theory enables the computation of eigenmodes within clusters of scatterers, shedding light on their dynamic responses to external stimuli.

Moreover, this document examines the practical implementation of multiple scattering theory in the context of specific geometries and material properties. We discuss its application in scenarios involving thin elastic plates, semi-infinite substrates, and other complex structures, highlighting its adaptability across diverse scenarios.

Furthermore, we delve into the challenges and limitations associated with numerical methods commonly used in structural analysis. By contrasting these with the advantages offered by multiple scattering theory, we underscore its efficacy in providing accurate and insightful solutions, particularly in scenarios where numerical methods may fall short.

Additionally, we showcase the practical utility of multiple scattering theory through case studies and demonstrations. We present examples of its application in resonance splitting phenomena and the design of optimal structural configurations, illustrating its potential to address real-world engineering challenges.

4.2. Relation to Other Project Documents

The present document is related with the following documents:

- Deliverable 1.1 T-matrix of spheres and pillars

5. Formulation of multiple scattering

The enclosed document offers a comprehensive derivation of the equations of multiple scattering theory tailored to the specific geometries and waves pertinent to our project. While these geometries may undergo adjustments in the upcoming months, the formulated framework remains versatile and applicable to any wave type and scatterer that may arise in the future.

D1.2 Formulation of multiple scattering

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Abstract

This document will state the theoretical formulation of the multiple scattering interaction between point-like scatterers in mechanical systems, both in thin elastic plates and in thick substrates. Moreover, we will discuss some possibilities for fastening the computation of eigenmodes and eigenstates in phononic architectures via this multiple scattering theory.

I. INTRODUCTION

Multiple scattering theory is a reliable method for calculating how clusters of scatterers respond to an incident field, whether they're finite or infinite. It can also compute the eigenmodes of these clusters, making it useful in many scenarios.

Despite the abundance of numerical tools available for solving such problems quickly, they have limitations. These tools often work like "black boxes," hiding the internal methods used for computation. This can leave users unsure about the accuracy of results, especially with complex structures. Additionally, handling complex scenarios such as large disordered clusters or finite structures within infinite domains requires significant computational resources, even with high-performance computers. Relying solely on numerical solutions also limits our ability to understand the underlying physical mechanisms due to the inability to manipulate mathematical expressions.

To address these challenges, multiple scattering theory offers a fast and effective analytical method for understanding complex structures. While perfect analytical solutions may not always be practical, approximations and simplified models tailored to the real system can improve understanding. Multiple scattering theory can thus complement existing commercial software.

For our project, multiple scattering theory is a valuable tool for designing and refining key structures before laboratory testing. This document summarizes the theory's key aspects and the various approximations used for specific problems, meeting the project's unique needs.

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II. MULTIPLE SCATTERING OF WAVES FOR FLEXURAL WAVES IN THIN ELASTIC PLATES

In this section we consider clusters of point-like scatterers on a 2-D thin plate. Let us consider an elastic plate of thickness h , mass density ρ , Young's modulus E and Poisson ratio ν , loaded with a finite distribution of scatterers located at positions $\mathbf{R}_\alpha$. According to Kirchhoff-Love theory, the governing equation of motion for the plate's out-of-plane displacement W in time-harmonic regime $\exp(-i\omega t)$ is given by [1]:

$$\mathcal{P}(\omega)W(\mathbf{r}) = (\nabla^4 - k^4 - \mathcal{R}(\omega)) W(\mathbf{r}) = 0, \quad (1)$$

where we introduced the biharmonic operator ∇^4 , the wave number k satisfying $k^4 = \omega^2 \rho h / D$, the plate bending stiffness $D = Eh^3 / 12 (1 - \nu^2)$ and the operator $\mathcal{R}(\omega)$ represents the distribution of inhomogeneities or scatterers in the cluster. This function can represent distributions of pinned points [2], masses [3], spring-mass resonators [1], and it is defined as:

$$\mathcal{R}(\omega)W(\mathbf{r}) = \sum_{\alpha} t_{\alpha}(\omega)W(\mathbf{R}_{\alpha})\delta(\mathbf{r} - \mathbf{R}_{\alpha}).$$

The quantities t_{α} are the resonators' strength or impedance.

Solving equation (1) can be done by first finding the Green's function of the plate without resonators satisfying

$$(\nabla^4 - k^4) G(\mathbf{r}) = \delta(\mathbf{r}),$$

The solution is known explicitly as:

$$G(\mathbf{r}) = \frac{i}{8k^2} [H_0(kr) - H_0(ikr)] \quad (2)$$

where H_0 is the zeroth-order Hankel function of the first kind.

For an incident excitation $\psi^i(\mathbf{r})$, the multiple scattering problem is solved by setting up a system of self-consistent equations, with the solution for the displacement $W(\mathbf{r})$ given by:

$$W(\mathbf{r}) = \psi^i(\mathbf{r}) + \sum_{\alpha} \phi_{\alpha} G(\mathbf{r} - \mathbf{R}_{\alpha}) \quad (3)$$

with the so-called "external" field $\psi^e(\mathbf{R}_{\alpha})$ representing the incident field on scatterer α , $\phi_{\alpha} = T_{\alpha}(\omega)\psi^e(\mathbf{R}_{\alpha})$. The external field is

$$\psi^e(\mathbf{R}_{\alpha}) = \psi^i(\mathbf{R}_{\alpha}) + \sum_{\beta \neq \alpha} \phi_{\beta} G(\mathbf{r} - \mathbf{R}_{\alpha}). \quad (4)$$

The coefficients T_α can be obtained from the response of a single scatterer, in which $\psi^e(\mathbf{R}_\alpha) = \psi^i(\mathbf{R}_\alpha)$ and the field is

$$W(\mathbf{r}) = \psi^i(\mathbf{r}) + T_\alpha \psi^i(\mathbf{R}_\alpha) G(\mathbf{r} - \mathbf{R}_\alpha). \quad (5)$$

Inserting this last equation into equation (1) and rearranging terms we can solve for T_α ,

$$T_\alpha = \frac{t_\alpha}{1 - it_\alpha / (8k^2)} \quad (6)$$

One finally obtains the system of equations in matrix form

$$M\Phi = \Psi^i, \quad (7)$$

with the elements of the matrix given by $M_{\alpha\beta} = \delta_{\alpha\beta} t_\alpha^{-1} - G(\mathbf{R}_\alpha - \mathbf{R}_\beta)$ and the right hand side is a vector containing the values of the incident field at the resonators' positions $\Psi_\alpha^i = \psi^i(\mathbf{R}_\alpha)$. This approach results in a system of N equations in N unknowns. The quantity t_α represents the strength of the α^{th} scatterer and it is characteristic of the physical properties of the scatterer.

This model of multiple scattering is useful but cannot contemplate other resonances such as bending moments, which induce not only displacement but moments on the surface of the plate. In [4], they have developed the multiple scattering equations for both compression modes and bending modes of a pillar on top of a thin elastic plate considering that the pillars are governed by Euler-Bernoulli theory and that their action on the plate is point-like. The singularities that appear from considering the moments on top of the plate are solved by regularising the Green's function and solving the asymptotic cases when we want to compute the field near the scatterer.

III. MULTIPLE SCATTERING THEORY OF POINT-LIKE SCATTERERS ON RAYLEIGH WAVES

The displacement vector will be in general expanded in three modes propagating along the xy plane with wavenumber $\mathbf{K}_G = \mathbf{K} + \mathbf{G}$, being the difference between them the component of the wavenumber in the z direction, which is $s\beta_G^\ell$ for $\ell = L, P, S$ and $s = +, -$. The polarization L is parallel to the three-dimensional longitudinal wave vector $\mathbf{k}_G^L = \mathbf{K}_G + s\beta_{fG}^L \hat{\mathbf{z}}$, while the P and S are perpendicular to the transverse wave vector $\mathbf{k}_G^T = \mathbf{K}_G + s\beta_{fG}^T \hat{\mathbf{z}}$,

with S being contained in the xy plane and perpendicular to P completing the orthogonal set. These polarization vectors are computed as

$$\mathbf{u}_G^L = \frac{1}{k_L} \mathbf{k}_G^L = \frac{1}{k_L} (\mathbf{K}_G + \beta_G^L \hat{\mathbf{z}}) \quad (8)$$

$$\mathbf{u}_G^S = \frac{1}{|\mathbf{K}_G|} \hat{\mathbf{z}} \times \mathbf{K}_G = \frac{1}{|\mathbf{K}_G|} (-\hat{\mathbf{y}} \cdot \mathbf{K}_G \hat{\mathbf{x}} + \hat{\mathbf{x}} \cdot \mathbf{K}_G \hat{\mathbf{y}}) \quad (9)$$

$$\mathbf{u}_G^P = \frac{1}{k_T} \mathbf{k}_G^T \times \mathbf{u}_G^S = -\frac{\hat{\mathbf{x}} \cdot \mathbf{K}_G \beta_G^T}{k_T |\mathbf{K}_G|} \hat{\mathbf{x}} - \frac{\hat{\mathbf{y}} \cdot \mathbf{K}_G \beta_G^T}{k_T |\mathbf{K}_G|} \hat{\mathbf{y}} + \frac{|\mathbf{K}_G|}{k_T} \hat{\mathbf{z}} \quad (10)$$

Since we are looking for bounded modes at the surface $z = 0$, we set $s = +$ only and assume that the positive region of the z axis is oriented towards the bottom of the page. Given the periodicity of the problem, a summation in all the reciprocal vectors will be made, thus

$$\mathbf{u} = \sum_{\mathbf{G}, \ell} \mathbf{u}_G^\ell A_G^\ell T_G^\ell(z) e^{i(\mathbf{K} + \mathbf{G}) \cdot \mathbf{r}} \quad (11)$$

where $T_G^\ell(z) = e^{i\beta_G^\ell z}$.

The components of the stress tensor in the z direction will therefore be

$$\sigma_{zz} = i \sum_{\mathbf{G}, \ell} [\lambda(\mathbf{K} + \mathbf{G}) \cdot \mathbf{u}_G^\ell + (\lambda + 2\mu)\beta_G^\ell \hat{\mathbf{z}} \cdot \mathbf{u}_G^\ell] A_G^\ell T_G^\ell(z) e^{i(\mathbf{K} + \mathbf{G}) \cdot \mathbf{r}} \quad (12)$$

$$\sigma_{xz} = i\mu \sum_{\mathbf{G}, \ell} [\hat{\mathbf{x}} \cdot (\mathbf{K} + \mathbf{G}) \hat{\mathbf{z}} \cdot \mathbf{u}_G^\ell + \beta_G^\ell \hat{\mathbf{x}} \cdot \mathbf{u}_G^\ell] A_G^\ell T_G^\ell(z) e^{i(\mathbf{K} + \mathbf{G}) \cdot \mathbf{r}} \quad (13)$$

$$\sigma_{yz} = i\mu \sum_{\mathbf{G}, \ell} [\hat{\mathbf{y}} \cdot (\mathbf{K} + \mathbf{G}) \hat{\mathbf{z}} \cdot \mathbf{u}_G^\ell + \beta_G^\ell \hat{\mathbf{y}} \cdot \mathbf{u}_G^\ell] A_G^\ell T_G^\ell(z) e^{i(\mathbf{K} + \mathbf{G}) \cdot \mathbf{r}} \quad (14)$$

Since the resonators on the surface will make a force along the vertical direction only, the boundary conditions at $z = 0$ will be $\sigma_{xz} = \sigma_{yz} = 0$, therefore

$$\sigma_{xz} = \sum_{\ell} [\hat{\mathbf{x}} \cdot (\mathbf{K} + \mathbf{G}) \hat{\mathbf{z}} \cdot \mathbf{u}_G^\ell + \beta_G^\ell \hat{\mathbf{x}} \cdot \mathbf{u}_G^\ell] A_G^\ell = 0 \quad (15)$$

$$\sigma_{yz} = \sum_{\ell} [\hat{\mathbf{y}} \cdot (\mathbf{K} + \mathbf{G}) \hat{\mathbf{z}} \cdot \mathbf{u}_G^\ell + \beta_G^\ell \hat{\mathbf{y}} \cdot \mathbf{u}_G^\ell] A_G^\ell = 0 \quad (16)$$

Since $\hat{\mathbf{z}} \cdot \mathbf{u}_G^S = 0$, and $\mathbf{K}_G \cdot \mathbf{u}_G^S = 0$, if we multiply the first and second equations by $-\hat{\mathbf{y}} \cdot \mathbf{K}_G$ and $\hat{\mathbf{x}} \cdot \mathbf{K}_G$, respectively, and we add them, we find

$$A_G^S = 0 \quad (17)$$

what means that, as it is well known, shear waves cannot be bounded at a free surface. Similarly, if we multiply the first and second equations by $\hat{\mathbf{x}} \cdot \mathbf{K}_G$ and $\hat{\mathbf{y}} \cdot \mathbf{K}_G$, respectively,

and we add them, we find

$$[|\mathbf{K}_G|^2 \hat{\mathbf{z}} \cdot \mathbf{u}_G^P + \beta_G^P \mathbf{K}_G \cdot \mathbf{u}_G^P] A_G^P = - [|\mathbf{K}_G|^2 \hat{\mathbf{z}} \cdot \mathbf{u}_G^L + \beta_G^L \mathbf{K}_G \cdot \mathbf{u}_G^L] A_G^L \quad (18)$$

or, after some algebra,

$$A_G^P = -2 \frac{\beta_G^L |\mathbf{K}_G|}{|\mathbf{K}_G|^2 - \beta_G^{T2}} \frac{k_T}{k_L} A_G^L \quad (19)$$

Therefore, we have that, at $z = 0$,

$$\sigma_{zz}(\mathbf{r}) = i \sum_{\mathbf{G}} [(\lambda k_L + 2\mu \beta_G^L \hat{\mathbf{z}} \cdot \mathbf{u}_G^L) A_G^L + 2\mu \beta_G^P \hat{\mathbf{z}} \cdot \mathbf{u}_G^P A_G^P] e^{i\mathbf{K}_G \cdot \mathbf{r}} \quad (20)$$

and, using the previous equation

$$\sigma_{zz}(\mathbf{r}) = \frac{i}{k_L} \sum_{\mathbf{G}} \left[\lambda k_L^2 + 2\mu \beta_G^{L2} - 4\mu \beta_G^P \beta_G^L \frac{|\mathbf{K}_G|^2}{|\mathbf{K}_G|^2 - \beta_G^{T2}} \right] A_G^L e^{i\mathbf{K}_G \cdot \mathbf{r}} \quad (21)$$

As explained in the previous section, the periodic arrangement of resonators placed on the surface of the interface makes a force on it given by

$$\sigma_{zz}(\mathbf{r}) = \sum_{\mathbf{R}_n} \sum_{\alpha} t_{\alpha} \delta(\mathbf{r} - \mathbf{R}_{\alpha n}) u_z(\mathbf{R}_{\alpha n}) \quad (22)$$

with t_{α} being the impedance of the scatterer which will be in general a resonant quantity.

Using

$$u_z(\mathbf{R}_{\alpha n}) = \sum_{\mathbf{G}} (\hat{\mathbf{z}} \cdot \mathbf{u}_G^L A_G^L + \hat{\mathbf{z}} \cdot \mathbf{u}_G^P A_G^P) e^{i\mathbf{K}_G \cdot \mathbf{R}_{\alpha n}} = -\frac{k_T^2}{k_L} \sum_{\mathbf{G}} \left(\frac{\beta_G^L}{|\mathbf{K}_G|^2 - \beta_G^{T2}} \right) A_G^L e^{i\mathbf{K}_G \cdot \mathbf{R}_{\alpha n}} \quad (23)$$

we find, after multiplication by $\exp(-\mathbf{K}_{G'} \cdot \mathbf{r})$ and integration over the unit cell,

$$\frac{i}{k_L} \left[\lambda k_L^2 + 2\mu \beta_{G'}^{L2} - 4\mu \beta_{G'}^T \beta_{G'}^L \frac{|\mathbf{K}_{G'}|^2}{|\mathbf{K}_{G'}|^2 - \beta_{G'}^{T2}} \right] A_{G'}^L = -\frac{k_T^2}{k_L} \sum_{\alpha} \frac{t_{\alpha}}{\Omega_c} \sum_{\mathbf{G}} \left(\frac{\beta_G^L}{|\mathbf{K}_G|^2 - \beta_G^{T2}} \right) A_G^L e^{i(\mathbf{G} - \mathbf{G}') \cdot \mathbf{R}_{\alpha}} \quad (24)$$

We can now define

$$X_{\alpha} = \sum_{\mathbf{G}} \left(\frac{\beta_G^L}{|\mathbf{K}_G|^2 - \beta_G^{T2}} \right) A_G^L e^{i\mathbf{G} \cdot \mathbf{R}_{\alpha}} \quad (25)$$

and, after repeating the process found before, we arrive to the secular equation

$$\sum_{\beta} M_{\alpha\beta} X_{\beta} = 0 \quad (26)$$

with

$$M_{\alpha\beta} = \delta_{\alpha\beta} - t_{\alpha} G_{\alpha\beta} \quad (27)$$

and

$$G_{\alpha\beta} = \frac{ik_T^2}{\Omega_c} \sum_{\mathbf{G}} \frac{\frac{\beta_{\mathbf{G}}^L}{|\mathbf{K}_{\mathbf{G}}|^2 - \beta_{\mathbf{G}}^{T2}}}{\lambda k_L^2 + 2\mu\beta_{\mathbf{G}}^{L2} - 4\mu\beta_{\mathbf{G}}^T\beta_{\mathbf{G}}^L \frac{|\mathbf{K}_{\mathbf{G}}|^2}{|\mathbf{K}_{\mathbf{G}}|^2 - \beta_{\mathbf{G}}^{T2}}} e^{i\mathbf{G} \cdot (\mathbf{R}_{\alpha} - \mathbf{R}_{\beta})} \quad (28)$$

or, after some algebra,

$$G_{\alpha\beta} = \frac{ik_T^2}{\Omega_c} \sum_{\mathbf{G}} \frac{\beta_{\mathbf{G}}^L e^{i\mathbf{G} \cdot (\mathbf{R}_{\alpha} - \mathbf{R}_{\beta})}}{[(\lambda + 2\mu)k_L^2 - 2\mu|\mathbf{K}_{\mathbf{G}}|^2] (|\mathbf{K}_{\mathbf{G}}|^2 - \beta_{\mathbf{G}}^{T2}) - 4\mu|\mathbf{K}_{\mathbf{G}}|^2 \beta_{\mathbf{G}}^T \beta_{\mathbf{G}}^L} \quad (29)$$

A. Finite cluster of point scatterers

In the previous section we have considered the eigenmodes of periodic distributions of finite clusters of scatterers. While this is a straightforward approach to understand either single lattices or complex clusters, if a more rigorous approach wants to be developed to study waveguides or finite clusters, we need to work on the lattice sum in equation (29), since the multiple scattering matrix $M_{\alpha\beta}$ remains unchanged.

Then, if the cluster of resonators is used to form a waveguide along the x direction, we have to make unit cell along the y axis infinite, then the Brillouin zone collapses to zero and the reciprocal lattice vectors $G_y = 2m\pi/L_y$ become a continuous variable, thus we have to make the replacement [5]

$$\frac{1}{\Omega_c} \sum_{\mathbf{G}} F(K_x + G_x, K_y + G_y) \rightarrow \frac{1}{2\pi a} \sum_n \int_{-\infty}^{\infty} F(K_x + 2n\pi/a, k_y) dk_y. \quad (30)$$

Similarly, if we are interested in the eigenmodes of an isolated cluster, we can make the full unit cell infinite and then the right replacement is

$$\frac{1}{\Omega_c} \sum_{\mathbf{G}} F(K_x + G_x, K_y + G_y) \rightarrow \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(k_x, k_y) dk_x dk_y. \quad (31)$$

Details on this replacement are given in Appendix A

In the case of a finite cluster, periodic sum in equation (29) is transformed into a double integral in k_x and k_y :

$$G_{\alpha\beta} = \frac{ik_T^2}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\beta_{\mathbf{G}}^L e^{i\mathbf{G} \cdot (\mathbf{R}_{\alpha} - \mathbf{R}_{\beta})}}{[(\lambda + 2\mu)k_L^2 - 2\mu|K_G|^2] (|K_G|^2 - \beta_G^{T2}) - 4\mu|K_G|^2 \beta_G^T \beta_G^L} dk_x dk_y, \quad (32)$$

being $|K_G|^2 = k_x^2 + k_y^2$, $\beta_G^L = \sqrt{k_L^2 - k_x^2 - k_y^2}$, $\beta_G^T = \sqrt{k_T^2 - k_x^2 - k_y^2}$ and $\mathbf{G} = (k_x, k_y)$. We change to polar coordinates (k, θ) , and we apply the plane wave expansion in cylindrical coordinates:

$$e^{iG(R_\alpha - R_\beta)} = \sum_{n=-\infty}^{\infty} i^n J_n(G(R_\alpha - R_\beta)) e^{in\theta}. \quad (33)$$

Then, the integral of this term is:

$$\int_0^\infty \int_0^\infty \sum_{n=-\infty}^{\infty} i^n J_n(k(R_\alpha - R_\beta)) e^{in\theta} d\theta dk = \int_0^\infty J_0(k(R_\alpha - R_\beta)) dk \quad (34)$$

In polar coordinates: $|K_G|^2 = k^2$, $\beta_G^L = \sqrt{k_L^2 - k^2}$, $\beta_G^T = \sqrt{k_T^2 - k^2}$. Then, the previous integral is

$$G_{\alpha\beta} = \frac{ik_T^2}{(2\pi)^2} \int_0^\infty \frac{k \sqrt{k_L^2 - k^2} J_0(k(R_\alpha - R_\beta))}{[(\lambda + 2\mu)k_L^2 - 2\mu k^2](2k^2 - k_T^2) - 4\mu k^2 \sqrt{k_T^2 - k^2} \sqrt{k_L^2 - k^2}} dk \quad (35)$$

IV. QUASIMODES OF A FINITE CLUSTER

In both the previous sections (finite clusters of scatterers on top of a thin elastic plate or on top of a thick substrate), the resulting system of equations (equations 7 and 26) define a $N \times N$ nonlinear eigenvalue problem. If we look now for solutions of the governing equations without excitation, then we need to find the zeros of the determinant of M , and we will obtain the quasimodes of the system by looking at the null space of M .

$$\det M(\omega_n) = 0 \quad (36)$$

There are various methods to solve equation (36), which can be broadly separated into two families, contour integral techniques [6–8] or iterative methods [9]. Each type has its advantage and inconvenient, contour integral methods can compute all singularities within a given region of the complex plane but can be slow because they require computing integrals on a closed path accurately, whereas iterative methods are faster but require an initial guess.

In this section we will focus on linearizing the eigenvalue problem by Taylor expanding the terms around the resonant frequency of the resonators. We will start by considering that all the scatterers in our system have the same physical properties. Then, if we expand their impedance around their resonant frequency

$$t_\alpha^{-1}(\omega) \sim t_\alpha^{-1}(\omega_R) + \frac{\partial t_\alpha^{-1}(\omega_R)}{\partial \omega} \delta\omega + \mathcal{O}(\delta\omega^2) = \frac{\partial t_\alpha^{-1}(\omega_R)}{\partial \omega} \delta\omega + \mathcal{O}(\delta\omega^2), \quad (37)$$

due to the resonant nature of the scatterer. Moreover, the Green function,

$$G_{\alpha\beta}(\omega) \sim G_{\alpha\beta}(\omega_R) + \frac{\partial G_{\alpha\beta}(\omega_R)}{\partial \omega} \delta\omega + \mathcal{O}(\delta\omega^2). \quad (38)$$

Therefore, the $M(\omega)$ matrix can be written

$$M_{\alpha\beta} = \delta\omega \left(\frac{\partial t_\alpha^{-1}(\omega_R)}{\partial \omega} \delta_{\alpha\beta} - \frac{\partial G_{\alpha\beta}(\omega_R)}{\partial \omega} \right) - G_{\alpha\beta}(\omega_R), \quad (39)$$

So that the null space problem becomes a generalized linear eigenvalue problem

$$A\Phi = \delta\omega B\Phi, \quad (40)$$

with

$$A_{\alpha\beta} = G_{\alpha\beta}(\omega_R) \quad (41)$$

$$B_{\alpha\beta} = \frac{\partial t_\alpha^{-1}(\omega_R)}{\partial \omega} \delta_{\alpha\beta} - \frac{\partial G_{\alpha\beta}(\omega_R)}{\partial \omega}. \quad (42)$$

In addition, we note that poles $\omega_{T\alpha}^\pm$ of T_α occur when the denominator in Eq. (6) vanishes i.e. when $t_\alpha^{-1}(\omega_{T\alpha}^\pm) = G(0)$:

$$\omega_{T\alpha}^\pm / \omega_{R\alpha} = \pm \sqrt{1 - q_\alpha^2} - iq_\alpha \quad \text{with } q_\alpha = \frac{1}{16} \sqrt{\frac{k_{R\alpha} m_{R\alpha}}{\rho h D}} = \frac{k_{R\alpha} a}{16D} \frac{\omega_{p0}}{\omega_{R\alpha}},$$

with $\omega_{p0}^2 = D/\rho h a^2$ the fundamental frequency of the plate and a is a characteristic length of the system. Those are eigenfrequencies of the system when only the self-interacting contribution of the resonators are considered, i.e. neglecting the off diagonal terms in M . One remarks that if $q_\alpha \leq 1$, we have $\omega_{T\alpha}^\pm / \omega_{R\alpha} = \pm e^{i\theta_\alpha}$ where $\theta_\alpha = -\arcsin(q_\alpha)$: they are simply related to the corresponding resonant frequencies of the scatterers by a rotation in the complex plane of an angle θ_α or $\pi - \theta_\alpha$. If $q_\alpha > 1$, they are purely imaginary.

Two simulations will be shown to illustrate the method: a pair of resonators on top of a thin elastic plate and a random cluster of resonators on top of a thin elastic plane.

1. Pair of identical resonators

In the case of only two resonators in the plate, the determinant of the multiple scattering matrix can be developed analytically and an expression for the approximate modes in the complex frequency plane is obtained. In this subsection, we are going to assume that the Green function is almost constant,

$$G(\omega) \simeq G(\omega_{R_\alpha}). \quad (43)$$

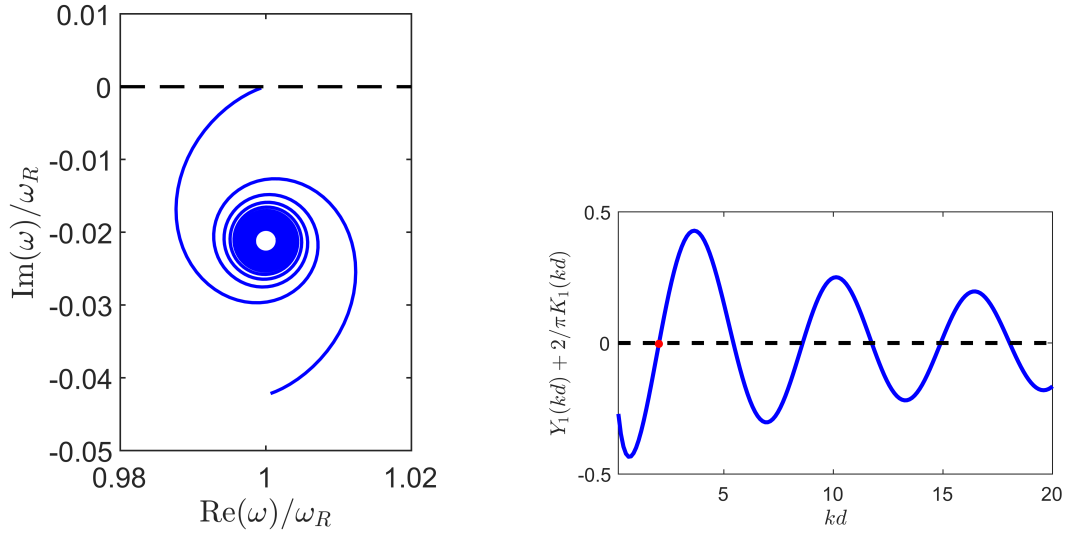
Then, equation (36) can be written as

$$\delta\omega = \frac{G(0)}{\partial_\omega t_0^{-1}} \pm \frac{G(d)}{\partial_\omega t_0^{-1}}, \quad (44)$$

which, after substitution of the Green's function and the impedance gives us

$$\delta\omega = -iq_\alpha \omega_{R_\alpha} (1 \pm \eta(d)), \quad (45)$$

where $\eta(d) = [H_0(k_R d) - 2i/\pi K_0(k_R d)]$.



(a) Solution space for a pair of resonators.

(b) Maximum splitting condition.

FIG. 1: Two scatterers with $\omega_{R_\alpha} = 100, m_{R_\alpha} = 0.001$ and $c_0 = 1$ in a thin elastic plate.

The trajectory of the modes in the complex frequency plane can be seen in Fig. (1a) as a function of the distance d between resonators. It can be seen that there is one single point

of the trajectory crossing the real frequency axis, which corresponds to $d = 0$. The center of the two spirals is given by

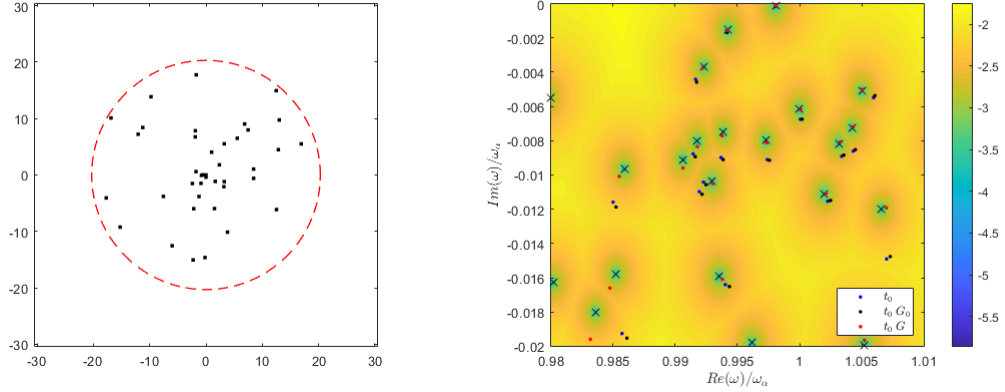
$$\omega_c = \omega_{R_\alpha}(1 - iq_\alpha). \quad (46)$$

It might be interesting to obtain the condition for the maximum frequency splitting between the two mods of this system (in-phase and out-of-phase). By deriving the real part of the frequency of the modes, we can obtain an expression for computing the maximum splitting between modes.

$$\partial_d \text{Re}(\omega_n) = 0 \rightarrow Y_1((\rho h/D)^{1/4} \sqrt{\omega_{R_\alpha}} d) + 2/\pi K_1((\rho h/D)^{1/4} \sqrt{\omega_{R_\alpha}} d) = 0. \quad (47)$$

This expression is represented in Fig. (1b).

2. Random cluster of identical resonators



(a) Resonators' cluster. The circular limit of $3\lambda_{R_\alpha}$ is indicated by the red dashed line. (b) Quasimodes in the complex plane for a random cluster of scatterers.

FIG. 2: Random cluster quasimode prediction for resonators with $\omega_{R_\alpha} = 0.93$, $m_{R_\alpha} = 0.1$ and $c_0 = 1$ in a thin elastic plate.

Figure 2b shows the prediction of the quasimodes of a finite cluster of resonators (40) on top of a thin elastic plate. The map shows the minimum eigenvalue of the multiple scattering matrix. The scatterers have all the same physical properties, and they have been located randomly within a circle of radius three times the resonating wavelength of the scatterers

($R_c = 3\lambda_{R_\alpha}$). Three predictions for the quasimodes have been performed, indicated by blue, black and red dots. The first one only develops as a Taylor expansion the impedance of the scatterers. The second one includes also the Green's function at the origin (diagonal term of the M matrix). Finally, the red dots expand the whole matrix. It can be seen that the third approach gives better results for the prediction of the quasimodes, being the difference more evident when $\text{Re}(\omega)/\omega_{R_\alpha} \neq 1$.

V. SUMMARY

In essence, multiple scattering theory continues to serve as a robust methodology for computing the eigenmodes of clusters of scatterers situated on elastic surfaces. While its foundational formulation, centered on point-like scatterers, yields only an approximate solution, it nevertheless provides invaluable insights for dissecting and comprehending complex structural arrangements.

The problem addressed within this document pertains to thin elastic plates and semi-infinite substrates, representing the primary geometries of interest in our ongoing project phase. Importantly, the methodology readily extends to encompass other geometries, necessitating solely an acquaintance with the Green's function. Moreover, the meticulous modeling of individual scatterer responses remains pivotal, as highlighted in Deliverable 1.1 of our project.

As a testament to its efficacy, we've devised an alternative solution method tailored to the resonance splitting phenomenon observed in closely spaced individual scatterers. This approach entails conducting a Taylor expansion around a resonant frequency, elucidating how collective behavior manifests in the spectral splitting of said frequency. Given our project's objective of identifying structures boasting numerous modes within a confined frequency range, this methodology assumes paramount importance. Consequently, we intend to persist in harnessing this technique for crafting optimal structural designs.

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Appendix A: Finite cluster series replacement

Let us assume we have a lattice sum of the form

$$S = \frac{1}{\Omega_C} \sum_{\mathbf{G}} f(\mathbf{K} + \mathbf{G}) \quad (\text{A1})$$

we are interested in the limit in which the unit cell becomes infinite and we study a finite structure. In this case, the unit cell is a square unit cell of lattice constant L , therefore the reciprocal lattice vectors are simply

$$\mathbf{G} = \frac{2\pi n}{L} \hat{\mathbf{x}} + \frac{2\pi m}{L} \hat{\mathbf{y}} \quad (\text{A2})$$

and the area of the unit cell Ω_C is

$$\Omega_C = L^2 \quad (\text{A3})$$

Also, the wavevector $\mathbf{K}$ can take values only within the Brillouin zone, so that

$$K_x \in [-\pi/L, \pi/L] \quad (\text{A4a})$$

$$K_y \in [-\pi/L, \pi/L] \quad (\text{A4b})$$

Now we wonder what happens with the sum S in the limit of $L \rightarrow \infty$. The first consequence is that, according to the above equations, the Brillouin zone collapses, so that the original wavevector $\mathbf{K}$ can take only the value 0. It will disappear from the sum S . However, the sum runs over all possible n and m values in the definition of the reciprocal lattice vector $\mathbf{G}$, so that the ratio $2\pi n/L$ and $2\pi m/L$ in this limit becomes a continuous variable. Thus we can write

$$\mathbf{K} + \mathbf{G} \rightarrow \frac{2\pi n}{L} \hat{\mathbf{x}} + \frac{2\pi m}{L} \hat{\mathbf{y}} \rightarrow k_x \hat{\mathbf{x}} + k_y \hat{\mathbf{y}} = \mathbf{k} \quad (\text{A5})$$

where, this new vector $\mathbf{k}$ is not the previous Bloch's wavevector $\mathbf{K}$, since this last one has collapsed due to the cancellation of the Brillouin zone. In the lattice sum we have now a continuous variable, however this sum is divided by the area of the unit cell, which is infinite, then we can write

$$\frac{1}{\Omega_C} = \frac{1}{L^2} = \frac{1}{(2\pi)^2} \frac{2\pi}{L} \frac{2\pi}{L} = \frac{1}{(2\pi)^2} dk_x dk_y \quad (\text{A6})$$

since, clearly, in the limit $L \rightarrow \infty$, $2\pi/L$ is the infinitesimal increment of both k_x and k_y .

Then we have

$$\lim_{L \rightarrow \infty} \frac{1}{\Omega_C} \sum_{\mathbf{G}} f(\mathbf{K} + \mathbf{G}) = \frac{1}{(2\pi)^2} \iint f(\mathbf{k}) d\mathbf{k} \quad (\text{A7})$$